MMP Learning Seminar

Week 100

Content:

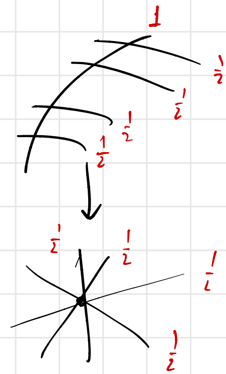
Coregularity of Fano varieties.

Regularity of a log pair: (X, Δ) normal & qp.

$$\operatorname{reg}(X,\Delta) = \max\left\{ k \;\middle|\; \begin{array}{l} \text{There exists a log resolution } (Y,\Delta_Y) \xrightarrow{\varphi} (X,\Delta) \\ \text{and prime divisors } S_1,\dots,S_k \subseteq \Delta_Y^{\geq 1} \text{ such that} \\ S_1 \cap \dots \cap S_k \neq \emptyset \end{array} \right\}$$

$-1.$

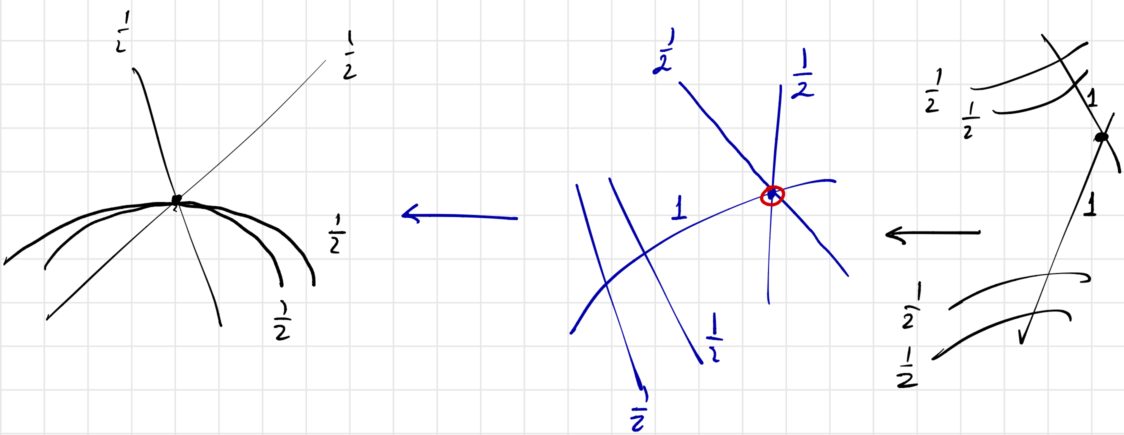
$$\varphi^*(K_X+\Delta) = K_Y + \Delta_Y.$$



Example: $(\mathbb{P}^2, \frac{1}{2}L_1 + \dots + \frac{1}{2}L_4)$

$$\operatorname{reg}(\mathbb{P}^2, \tfrac{1}{2}(\textstyle\sum_{i=1}^{4} L_i)) = 0.$$

$(\mathbb{P}^2, \frac{1}{2}L_1 + \frac{1}{2}L_2 + \frac{1}{2}C_1 + \frac{1}{2}C_2)$ $\qquad \operatorname{reg} = 1.$

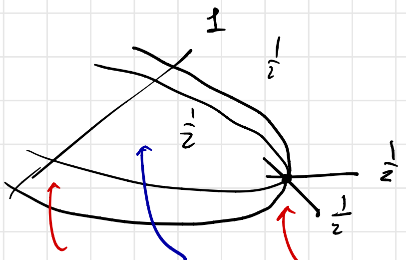


Remark: For a n-dim pair, the reg is in $\{-1,\dots,n-1\}$.

all of them passing through $[0:\dots:0:1]$

$$\left(\mathbb{P}^2, \frac{1}{n}\underbrace{(L_1+\dots+L_n)}_{\text{are transversal.}} + \frac{1}{n}\underbrace{(C_1+\dots+C_n)}\right)$$

All of them have the same tangent direction which is different from tangent dir of lines.



reg = 1.

Absolute regularity: X normal projective.

$$\hat{\operatorname{reg}}(X) = \max\left\{ \operatorname{reg}(X,\Delta) \;\middle|\; (X,\Delta) \text{ is } \underset{\substack{\| \\ K_X+\Delta\equiv 0. \\ \& \\ (X,\Delta) \text{ is lc.}}}{\log \text{ CY}} \right\}$$

If $\{\ \}=\emptyset$, then $\hat{\operatorname{reg}}(X) = -\infty$.

Absolute coregularity:

$$\hat{\operatorname{coreg}}(X) = \underset{\substack{\| \\ n}}{\dim X} - \underbrace{\hat{\operatorname{reg}}\, X}_{-(n-1)} \underbrace{-\,1}_{-1}.$$

Example: $X \cong \mathbb{P}^n$, then $\hat{\operatorname{coreg}}(\mathbb{P}^n) = 0$.

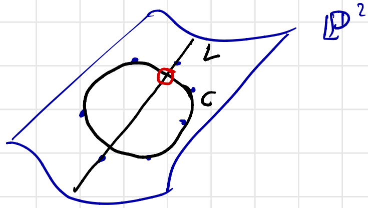
For a Fano variety $-K_X$ is ample so $\hat{\operatorname{coreg}}(X) \in \{0,\dots, \underset{\substack{\| \\ \dim X}}{n}\}$

All toric varieties belong to this class.

exceptional Fano varieties

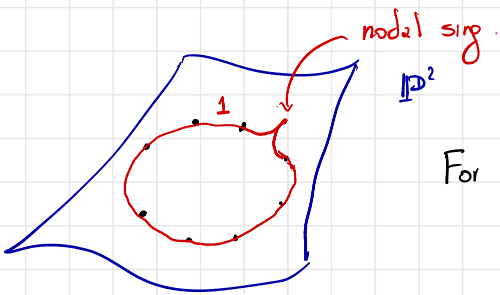
Examples:

- del Pezzo surfaces: Any del Pezzo surface of degree ≥ 2 has $\widehat{\mathrm{coreg}} = 0$.



$(\mathbb{P}^2, L + C)$

- del Pezzo surfaces of degree 1:



$\mathrm{coreg} \leq 1.$

For a general dP_1 the $\widehat{\mathrm{coreg}} = 0$.

What about higher dimensions:

There are 105 families of Fano terminal 3-folds.

Loginov: At least 100 of these families have general element of $\widehat{\mathrm{coreg}} = 0$.

Problem: Classify all Fano Gorenstein surfaces of $\widehat{\mathrm{coreg}} = 0$.

- Classification of Fano Gorenstein surfaces of $\rho = 1$. //

Complements:

Definition: A N-complement on X is a boundary Δ:

i) (X, Δ) is log CY &

ii) $N(K_X + \Delta) \sim 0$.

$N\Delta \in |-NK_X|$.

controlled multiple of the anti-canonical.

Example: X is a toric variety, then X admits a 1-comp of $\widehat{\text{coreg}} = 0$.

Theorem (Birkar, 2016): Let X be a n-dimensional Fano type var. Then, X admits a $N(n)$-complement.

Q: How fast does $N(n)$ grows?

$N(1) = 1$.

$N(2) = ??$ $\quad$ $N(2) = 66$.

exceptional Fano varieties.

Theorem (Totaro, 2022): There exists a Fano 4-fold which admits no m-complement for $m \leq 1.966.233$.

$N(4) \geq 1.966.233$.

Remark: $N(n)$ grows at least doubly exponentially with n.

X Fano of dimension n, $\hat{\mathrm{coreg}}(X) \in \{0, \ldots, n\}$

what happens here ?

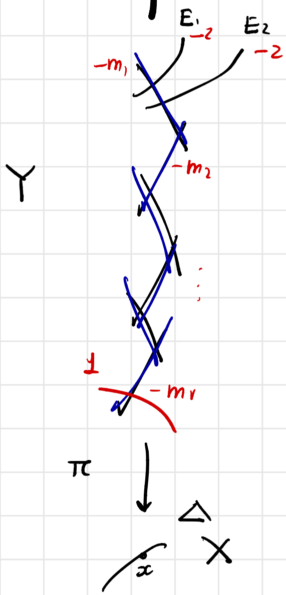
$N(n)$ grows fast

Theorem 1 (Figueroa - Filipazzi - M - Peng, 2022):
A Fano of $\hat{\mathrm{coreg}} = 0$ admits a 1-comp or 2-comp of $\hat{\mathrm{coreg}} = 0$.

Theorem 0 (Filipazzi - Mauri - M, 2023):
Let (X, Δ) be a log CY pair of $\mathrm{coreg} = 0$. Assume Δ has standard coefficients. Then $2(K_X + \Delta) \sim 0$.

Thm 0 is one of the tools used in the proof of Thm 1.

Examples: D-type singularities.



For D_n singularities, we have $m_1 = \ldots = m_r = 2$.

Q: Can we have a 1-complement.

$$a_{E_1}(X) = a_{E_2}(X) \leq \frac{1}{2}.$$

$a_F(X) < 1.$

(X, Δ) 1-complement.

Then $a_{E_1}(X,\Delta) = a_{E_2}(X,\Delta) = 0.$

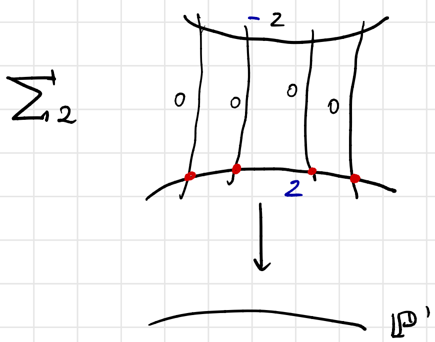
$a_F(X,\Delta) = 0$

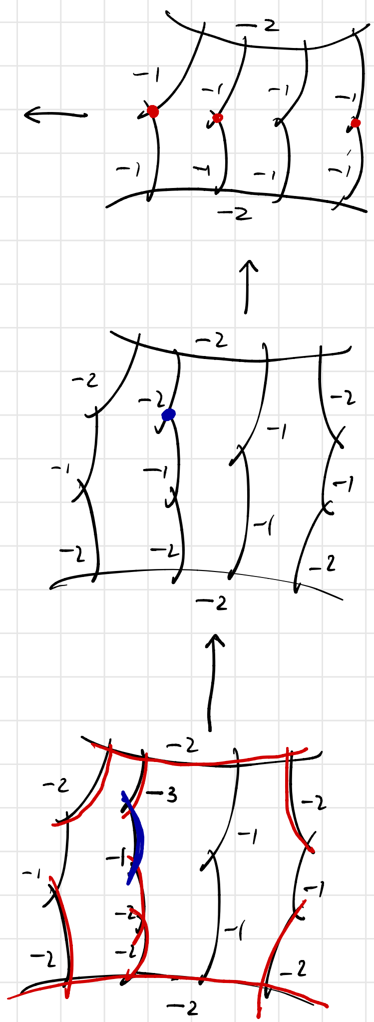
$\longrightarrow \longleftarrow$

No non-trivial (strict lc) 1-comp for D-type sing.

However, D-type sing admit reduced 2-complements.

Fano surface with $\rho = 1$ & no 1-comp of $\widehat{\mathrm{coreg}} = 0$.

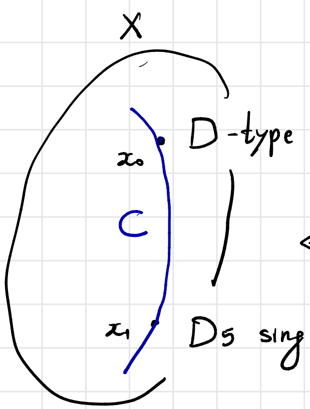




$\mathrm{mld}_{x_0}(X, \Delta) = 0$

$\mathrm{mld}_{x_0}(X) < 1$

$\mathrm{mld}_{x_1}(X) = 1$



(X, C) is log CY & $2(K_X + C) \sim 0$.

Sketch of the proof of Thm 1:

X Fano variety of $\hat{\mathrm{coreg}} = 0$, $\boxed{n = \dim X.}$

(X,Γ) a complement of $\mathrm{coreg}(X,\Gamma) = 0$.

dlt model. Y Fano type $\Longrightarrow$ Y is a MDS.

$$\begin{tikzcd} (Y,\Gamma_Y) \arrow[d, "\pi"'] \\ (X,\Gamma) \arrow[u, bend right, "X\ \mathrm{FT}"'] \end{tikzcd}$$

, $E_1,\dots,E_n \subseteq \lfloor \Gamma_Y \rfloor$ with $E_1 \cap \dots \cap E_n \neq \emptyset$.

$|-2(K_Y + E_1 + \dots + E_n)|$ ⟵ if we find a lc element here, then we are done.

$$\pi^*(K_X + \Gamma) = K_Y + \Gamma_Y$$

Run a $(-2(K_Y + E_1 + \dots + E_n))$-MMP.

$$\begin{tikzcd} Y \arrow[r, dashed] & Z \arrow[d] \\ & W \end{tikzcd}$$

$-2(K_Z + E_{Z,1} + \dots + E_{Z,n})$ is semiample.

ample model.

Using the negativity Lemma it suffices to produce a lc element in

$$|-2(K_Z + E_{Z,1} + \dots + E_{Z,n})|.$$

$$\begin{tikzcd} Y \arrow[r, dashed] \arrow[d] & Z \arrow[d] \\ X & W \end{tikzcd}$$

$|-2(K_Z + E_{Z,1} + \cdots + E_{Z,n})|$.

- $\dim W = 0$. $K_Z + \sum_{i=1}^n E_{Z,i} \equiv 0$ so by FMM22.
  $2(K_Z + \sum_{i=1}^n E_{Z,i}) \sim 0$.

- $\dim W = \dim X$. $-(K_Z + \sum_{i=1}^n E_{Z,i})$ big & semiample
  We can find a component, say E_1 s.t.

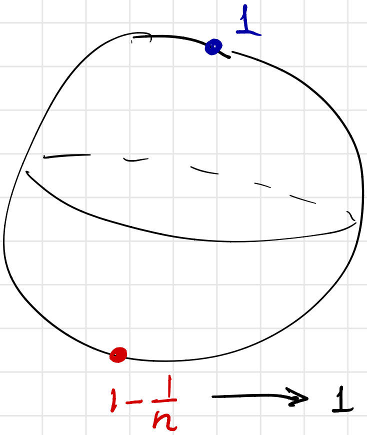
  $$H^0(-2(K_Z + \sum_{i=1}^n E_{Z,i})) \twoheadrightarrow H^0(-2(K_{E_1} + \sum_{i=2}^n E_i|_{E_1}))$$

  FT of coreg $=0$.
  However the coeff may become standard.

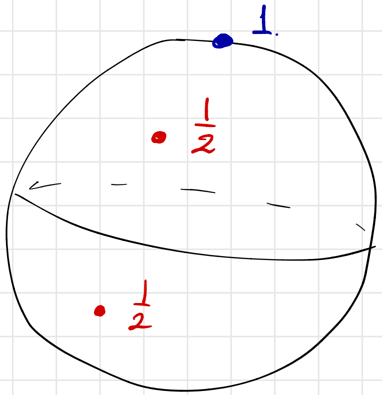
- $\dim W \in \{1, \ldots, \dim X - 1\}$, there is a component, E_1.
  such that E_1 is vertical is base

  Trick: perturb $E_2, \ldots, E_n$ so that $-(K_Z + E_1 + (1-\delta)\sum_{i=2}^n E_i)$
  is big & nef.

Only case left (dim 1): $\mathrm{coreg} = 0$



1 - complement.



2 - complement.

$\square$